TRANSFER OF R-GROUPS BETWEEN p-ADIC INNER FORMS OF SL_n

KWANGHO CHOIY AND DAVID GOLDBERG

ABSTRACT. We study the Knapp-Stein R–groups for inner forms of the split group $SL_n(F)$, with F a p–adic field of characteristic zero. Thus, we consider the groups $SL_m(D)$, with D a central division algebra over F of dimension d^2, and $m = n/d$. We use the generalized Jacquet-Langlands correspondence and results of the first named author to describe the zeros of Plancherel measures. Combined with a study of the behavior of the stabilizer of representations by elements of the Weyl group we are able to determine the Knapp-Stein R–groups in terms of those for $SL_n(F)$. We show the R–group for the inner form embeds as a subgroup of the R–group for the split form, and we characterize the quotient. We are further able to show the Knapp-Stein R–group is isomorphic to the Arthur, or Endoscopic R–group as predicted by Arthur. Finally, we give some results on multiplicities and actions of Weyl groups on L–packets.

CONTENTS

1. Introduction

We embark on a study of the non-discrete tempered spectrum of non-quasi-split inner forms of groups of classical types. Our approach is to use a generalized version of the Jacquet-Langlands correspondence to transfer information about the tempered spectrum of the split form to the inner form. Here we study inner forms of $G = SL_n(F)$, with F a p–adic field of characteristic zero. Any such inner form is of the form $G' = SL_m(D)$, where D is a central division algebra of degree d^2 over F, and $m = n/d$. Let $\widetilde{G} = GL_n(F)$ and $\widetilde{G}' = GL_m(D)$. Then $\widetilde{G}$ and $\widetilde{G}'$ are inner forms. If $\widetilde{P}$ is a parabolic subgroup of $\widetilde{G}$, then $P = \widetilde{P} \cap G$ is a parabolic subgroup of G, and every parabolic subgroup of G arises in this way. Similarly, if $\widetilde{P}'$ is a parabolic subgroup of $\widetilde{G}'$, then $P' = \widetilde{P} \cap G'$ is a parabolic subgroup of G', and all parabolic subgroups of G' arise in this way. Now, let $\widetilde{P}' = \widetilde{M}'\widetilde{N}'$ be the Levi decomposition of $\widetilde{P}'$. Then $\widetilde{M}' \simeq GL_{m_1}(D) \times \cdots \times GL_{m_k}(D)$, for some partition $m_1 + \cdots + m_k = m$. Then there is a corresponding parabolic subgroup $\widetilde{P} = \widetilde{M}\widetilde{N}$ of $\widetilde{G}$, with $\widetilde{M} = GL_{m_1 d}(F) \times \cdots \times GL_{m_k d}(F)$. Now let $M = \widetilde{M} \cap G$ and $M' = \widetilde{M}' \cap G'$. There is a generalized Jacquet-Langlands correspondence between L–packets on M and L–packets on M'. That is, since M and M' are inner forms, and

$$\prod_j SL_{m_j d}(F) \subset M \subset \prod_j GL_{m_j d}(F)$$

we have a description of the L–packets of M' in terms of those for M. Briefly, we see there is a Jacquet-Langlands correspondence from $\widetilde{M}$ to $\widetilde{M}'$, given by the product of the Jacquet-Langlands correspondences from $GL_{m_i d}(F)$ to $GL_{m_i}(D)$ [DKV84, Rog83]. Let $\widetilde{\sigma}$ be an irreducible discrete series representation of $\widetilde{M}$, and let

$$\phi : W_F \longrightarrow {}^L M = \prod_j GL_{m_j d}(\mathbb{C})$$

be the Langlands parameter given by [HT01, Hen00]. Let $\widetilde{\sigma}'$ be the representation corresponding to $\widetilde{\sigma}$ through the Jacquet-Langlands correspondence, and note ϕ is also the Langlands parameter for $\widetilde{\sigma}'$. Then using [GK82], we see the components of $\widetilde{\sigma}|_M$ form an L–packet of $\Pi_\phi(M)$, with Langlands parameter $\mathrm{pr} \circ \phi : W_F \to {}^L M$, where pr is the projection from $GL_n(\mathbb{C})$ to $PGL_n(\mathbb{C})$. Similarly, the components of $\widetilde{\sigma}'|_{M'}$ form an L–packet, $\Pi_\phi(M')$ of M', and we say $\Pi_\phi(M)$ and $\Pi_\phi(M')$ correspond by a generalized Jacquet-Langlands correspondence. In fact, such a correspondence always exists between discrete series of inner forms H and H' if

$$\prod_{j=i}^{k} SL_{\ell_j}(F) \subset H \subset \prod_{j=1}^{k} GL_{\ell_j}(F),$$

[Cho12b]. If $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ then we say σ and σ' correspond under this generalized Jacquet-Langlands type correspondence.

Let A be the split component of M, and A' the split component of M'. We denote the reduced roots by $\Phi(P, A) = \Phi(P', A')$. Let $W_M = W_{M'}$ be the Weyl group $N_G(A)/A$. We identify these two Weyl groups for the purpose of this exposition. The Knapp-Stein R–group, R_σ, along with a 2–cocycle, determines the structure of the (normalized) induced representation $i_{GM}(\sigma)$, and similarly we have the Knapp-Stein R–group $R_{\sigma'}$ attached to $i_{G'M'}(\sigma')$. The R–group, R_σ can be realized as a

quotient of two subgroups of W_M. The first is the stabilizer $W(\sigma)$ of σ in W_M. The second, W'_σ is generated by the root reflections in the zeros of the rank 1 Plancherel measures. By the results of [Cho12b], we have the transfer of the Plancherel measures (cf. Proposition 4.11 and its proof). In particular, for any $\beta \in \Phi(P, A)$, we have $\mu_\beta(\sigma) = \mu_\beta(\sigma')$. This shows $W'_\sigma = W'_{\sigma'}$. Thus, it is left to describe $W(\sigma)$. In [Gol94a] the second named author showed

$$(1.1) \qquad W(\sigma) = \left\{ w \in W_M |{}^w \widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta \text{ for some character } \eta \in \left(\widetilde{M}/M \right)^D \right\},$$

where the superscript D indicates the Pontrjagin dual. As in [Sha83, Gol94a] it is straightforward to see

$$(1.2) \quad W(\sigma') \subset \left\{ w \in W_{M'} |{}^w \widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta \text{ for some character } \eta \in \left(\widetilde{M}'/M' \right)^D \right\}.$$

The proof of equality of equation (1.1) in [Gol94a] relied on multiplicity one of restriction from $GL_n(F)$ to $SL_n(F)$. We know this multiplicity one property fails for restriction from $GL_m(D)$ to $SL_m(D)$, [HS11], It is straightforward to see the right hand sides of equations (1.1) and (1.2) are equal (cf. Proposition 4.9 and its proof) under the identification of W_M and $W_{M'}$. Thus, $W(\sigma') \subset W(\sigma)$, and hence $R_{\sigma'} \subset R_\sigma$ (cf. Theorem 4.13). We also give a characterization of the quotient $R_\sigma/R_{\sigma'}$ (cf. Remark 4.17).

We note the study of R–groups is crucial to understanding the elliptic tempered spectrum of reductive groups [Art93]. Further, the isomorphism of Knapp-Stein R–groups and the Arthur R groups play an important role in the trace formula, and in particular in the transfer of automorphic forms [Art11].

In Section 2 we recall the background information we need on reducibility and R–groups. We also discuss the structure of inner forms and specify the results we need to $SL_n(F)$ and its inner forms. We describe the generalized Jacquet-Langlands correspondence, as described in [Cho12b]. In Section 3 we describe the elliptic L–packets of Levi subgroups of $SL_n(F)$ and its inner forms. In Section 4 we prove our main results on the structure of R–groups for the inner forms of $SL_n(F)$. Finally, in Section 5 we discuss some results on multiplicity which arise for the inner forms of $SL_n(F)$. In an appendix also give a description of the action of characters of M and M' on the L–packets of these two groups.

As we finalized this manuscript we noted a preprint by K.F. Chao and W.W. Li [CL12], posted to the Arxiv a few days prior, which addresses the same problem. We note our results are derived independently and based on the work of the first named author on the transfer of Plancherel measures via the generalized Jacquet-Langlands correspondence, which is a different approach than in [CL12], where they work mostly with the R–groups on the dual side. They are able to give a description of the cocyle η and in particular give examples where η is non-trivial. This gives examples of induced representations with abelian R–groups which decompose with multiplicity greater than one. We also point out example 6.3.4 of [CL12] shows the containment $R_{\sigma'} \subset R_\sigma$ can be proper.

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communications and discussions which improved the quality of the results in this manuscript.

2. PRELIMINARIES

In this section, we recall background materials and review known facts.

2.1. **Notation.** Throughout this paper, F denotes a p-adic field of characteristic 0, that is, a finite extension of $\mathbb{Q}_p$, with an algebraic closure $\bar{F}$. Let $\mathbf{G}$ be a connected reductive group over F. We let $G = \mathbf{G}(F)$ and use similar notation for other algebraic groups. Fix a minimal F-parabolic subgroup $\mathbf{P}_0$ of G with Levi component $\mathbf{M}_0$ and unipotent radical $\mathbf{N}_0$. Let $\mathbf{A}_0$ be the split component of $\mathbf{M}_0$, that is, the maximal F-split torus in the center of $\mathbf{M}_0$. Let Δ be the set of simple roots of $\mathbf{A}_0$ in $\mathbf{N}_0$. Let $\mathbf{P} \subseteq \mathbf{G}$ be a standard (that is, containing $\mathbf{P}_0$) F-parabolic subgroup of $\mathbf{G}$. Write $\mathbf{P} = \mathbf{MN}$ with its Levi component $\mathbf{M} = \mathbf{M}_\theta \supseteq \mathbf{M}_0$ generated by a subset $\theta \subseteq \Delta$ and its unipotent radical $\mathbf{N} \subseteq \mathbf{N}_0$. We denote by δ_P the modulus character of P.

Let $\mathbf{A_M}$ be the split component of $\mathbf{M}$. Denote by $X^*(\mathbf{M})_F$ the group of F-rational characters of $\mathbf{M}$. We denote by $\Phi(P, A_M)$ the reduced roots of $\mathbf{P}$ with respect to $\mathbf{A_M}$. Denote by $W_\mathbf{M} = W(\mathbf{G}, \mathbf{A_M}) := N_\mathbf{G}(\mathbf{A_M})/Z_\mathbf{G}(\mathbf{A_M})$ the Weyl group of $\mathbf{A_M}$ in $\mathbf{G}$, where $N_\mathbf{G}(\mathbf{A_M})$ and $Z_\mathbf{G}(\mathbf{A_M})$ are respectively the normalizer and the centralizer of $\mathbf{A_M}$ in $\mathbf{G}$. For simplicity, we write $\mathbf{A}_0 = \mathbf{A_{M_0}}$.

By abuse of terminology, we make no distinction between the set of isomorphism classes with the set of representatives. Let $\mathrm{Irr}(M)$ denote the set of isomorphism classes of irreducible admissible representations of $M = \mathbf{M}(F)$. For any $\sigma \in \mathrm{Irr}(M)$, we write $i_{GM}(\sigma)$ for the normalized (twisted by $\delta_P^{1/2}$) induced representation. Denote by $\Pi_{\mathrm{disc}}(M)$ the set of discrete series representations of M.

We denote by $H^i(F, \mathbf{G}) := H^i(\mathrm{Gal}(\bar{F}/F), \mathbf{G}(\bar{F}))$ the (nonabelian) Galois cohomology of $\mathbf{G}$. Denote by W_F the Weil group of F and $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$. Fixing Γ_F-invariant splitting data, we define the Langlands dual (L-group) of G as a semi-direct product $^L G := \widehat{G} \rtimes \Gamma_F$ (see [Bor79, Section 2]). For any topological group H, we denote by $\pi_0(H)$ the group H/H° of connected components of H, where H° denotes the identity component of H. By $Z(H)$ we will denote the center of H. We write $(H)^D$ for the group $\mathrm{Hom}(H, \mathbb{C}^\times)$ of all continuous characters. We say a character is unitary if its image is in the unit circle $S^1 \subset \mathbb{C}^\times$.

For a central division algebra D over F, we let $GL_m(D)$ denote the group of all invertible elements of $m \times m$ matrices over D, and let $SL_m(D)$ be the subgroup of elements in $GL_m(D)$ whose reduced norm is 1 (see [PR94, Sections 1.4 and 2.3]).

For any finite set S, we write $|S|$ for the cardinality of S. For two integers a and b, $a|b$ means that b is divisible by a.

2.2. **R-groups.** For $\sigma \in \mathrm{Irr}(M)$ and $w \in W_M$, we let $^w\sigma$ be the representation given by $^w\sigma(x) = \sigma(w^{-1}xw)$. Given $\sigma \in \Pi_{\mathrm{disc}}(M)$, we define

$$W(\sigma) := \{w \in W_M : {}^w\sigma \simeq \sigma\}.$$

Set $\Delta'_\sigma = \{\beta \in \Phi(P, A_M) : \mu_\beta(\sigma) = 0\}$, where $\mu_\beta(\sigma)$ is the Plancherel measure attached to σ [Gol94b, p.1108]. Denote by W'_σ the normal subgroup of $W(\sigma)$ generated by the reflections in the roots of Δ'_σ. *The Knapp-Stein R-group* is defined by

$$R_\sigma := \{w \in W(\sigma) : w\beta > 0, \ \forall \beta \in \Delta'_\sigma\}.$$

We write $C(\sigma) := \mathrm{End}_G(i_{GM}(\sigma))$ for the algebra of G-endomorphisms of $i_{GM}(\sigma)$. We call it the commuting algebra of $i_{GM}(\sigma)$.

Theorem 2.1 (Knapp-Stein [KS72]; Silberger [Sil78, Sil79]). *For any $\sigma \in \Pi_{\mathrm{disc}}(M)$, we have*

$$W(\sigma) = R(\sigma) \ltimes W'_\sigma.$$

Moreover, $C(\sigma) \simeq \mathbb{C}[R(\sigma)]_\eta$, the group algebra of $R(\sigma)$ twisted by a 2-cocycle η, which is explicitly defined in terms of group $W(\sigma)$.

Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ be an L-parameter. We denote by $C_\phi(\widehat{M})$ the centralizer of the image of ϕ in $\widehat{M}$ and by $C_\phi(\widehat{M})^\circ$ its identity component. Fix a maximal torus T_ϕ in $C_\phi(\widehat{M})^\circ$. We define

$$W_\phi^\circ := N_{C_\phi(\widehat{M})^\circ}(T_\phi)/T_\phi, \ W_\phi := N_{C_\phi(\widehat{M})}(T_\phi)/T_\phi \text{ and } R_\phi := W_\phi/W_\phi^\circ.$$

Note that W_ϕ can be identified with a subgroup of W_M (see [Art89, p.45]). Let $\Pi_\phi(M)$ be the L-packet associated to the L-parameter ϕ. For $\sigma \in \Pi_\phi(M)$, we set

$$W_{\phi,\sigma}^\circ := W_\phi^\circ \cap W(\sigma), \ W_{\phi,\sigma} := W_\phi \cap W(\sigma) \text{ and } R_{\phi,\sigma} := W_{\phi,\sigma}/W_{\phi,\sigma}^\circ.$$

We call $R_{\phi,\sigma}$ the *Arthur R-group*.

Conjecture 2.2. *Let $\sigma \in \Pi_\phi(M)$ be a discrete series representation. Then we have*

$$R_\sigma \simeq R_{\phi,\sigma}.$$

2.3. **Inner Forms.** Let $\mathbf{G}$ and $\mathbf{G}'$ be connected reductive groups over F. We say that $\mathbf{G}$ and $\mathbf{G}'$ are F-*inner forms* with respect to an $\bar{F}$-isomorphism $\varphi : \mathbf{G}' \xrightarrow{\sim} \mathbf{G}$ if $\varphi \circ \tau(\varphi)^{-1}$ is an inner automorphism ($g \mapsto xgx^{-1}$) defined over $\bar{F}$ for all $\tau \in \mathrm{Gal}(\bar{F}/F)$ (see [Shi10, p.851]). If there is no confusion, we often omit the references to F and φ. Set $\mathbf{G}^{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$. We note [Kot97, p.280] that there is a bijection between $H^1(F, \mathbf{G}^{\mathrm{ad}})$ and the set of isomorphism classes of F-inner forms of $\mathbf{G}$.

Suppose $\mathbf{G}$ is either GL_n or SL_n over F. Then the set of isomorphism classes of F-inner forms of $\mathbf{G}$ is in bijection with the subgroup $Br(F)_n$ of n-torsion elements in the Brauer group $Br(F)$ (see [Cho12b, Section 2.3]). Hence the group G' of F-rational points of any F-inner form $\mathbf{G}'$ of GL_n (respectively, SL_n) is of the form $GL_m(D)$ (respectively, $SL_m(D)$) where D is a central division algebra of dimension d^2 over F and $n = md$.

2.4. **Structure of Levi Subgroups of SL_n and its inner forms.** Let $\widetilde{\mathbf{G}}$ be GL_n over F. Let $\widetilde{\mathbf{P}}_0$ be the minimal F–parabolic subgroup of upper triangular matrices in $\widetilde{\mathbf{G}}$. Set the minimal F-Levi subgroup $\widetilde{\mathbf{A}}_0 = \widetilde{\mathbf{M}}_0$ to be the group of diagonal matrices, and set the unipotent radical $\widetilde{\mathbf{N}}_0$ to be the group of unipotent upper triangular matrices. Let $\widetilde{\Delta}$ be the simple roots of $\widetilde{\mathbf{A}}_0$ in $\widetilde{\mathbf{N}}_0$, so $\widetilde{\Delta} = \{e_i - e_{i+1} : 1 \le i \le n-1\}$. Let $\widetilde{\mathbf{M}} = \widetilde{\mathbf{M}}_\theta$ be an F-Levi subgroup of $\widetilde{\mathbf{G}}$ for some subset $\theta \subseteq \Delta$. Then $\widetilde{M}$ is of the form $\prod_{i=1}^k GL_{n_i}(F)$ for some positive integer k and n_i.

Let $\widetilde{\mathbf{G}}'$ be an F-inner form of $\widetilde{\mathbf{G}}$, and let $\widetilde{\mathbf{M}}'$ be an F-Levi subgroup of $\widetilde{\mathbf{G}}'$ that is an F-inner form of $\widetilde{\mathbf{M}}$. Then $\widetilde{G}'$ is of the form $GL_m(D)$ for some central division algebra D of dimension d^2 over F with $n = md$, and $\widetilde{M}'$ is of the form $\prod_{i=1}^k GL_{m_i}(D)$ with $n_i = m_i d_i$.

Let $\mathbf{G}$ be SL_n over F. Let $\mathbf{P}_0 = \widetilde{\mathbf{P}}_0 \cap \mathbf{G}$ be our fixed minimal F–parabolic subgroup of $\mathbf{G}$. Set the minimal F-Levi subgroup $\mathbf{A}_0 = \mathbf{M}_0$ to be $\widetilde{\mathbf{M}}_0 \cap \mathbf{G}$, and set

the unipotent radical $\mathbf{N}_0$ to be $\widetilde{\mathbf{N}}_0 \cap \mathbf{G} = \widetilde{\mathbf{N}}_0$. We identify the simple roots Δ with $\widetilde{\Delta}$. Let $\mathbf{M}$ be an F-Levi subgroup of $\mathbf{G}$. Then we have
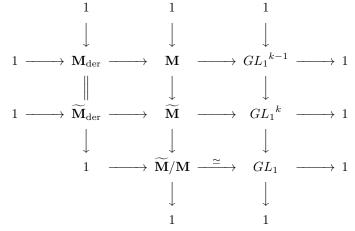
$$(2.1) \qquad \prod_{i=1}^{k} SL_{n_i} \subseteq \mathbf{M} \subseteq \prod_{i=1}^{k} GL_{n_i}.$$

Let $\mathbf{G}'$ be an F-inner form of $\mathbf{G}$, and Let $\mathbf{M}'$ be an F-Levi subgroup of $\mathbf{G}'$ that is an F-inner form of $\mathbf{M}'$ Then G' is of the form $SL_m(D)$, and M' is of the form

$$(2.2) \qquad \prod_{i=1}^{k} SL_{m_i}(D) \subseteq M' \subseteq \prod_{i=1}^{k} GL_{m_i}(D).$$

The Weyl groups W_M, $W_{\widetilde{M}}$, $W_{M'}$ and $W_{\widetilde{M}'}$ can be all identified and realized as a subgroup of the group S_k of permutations on k letters.

Remark 2.3. We have the following commutative diagram of algebraic groups:

$$
\begin{array}{ccccccccc}
& & 1 & & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{M}_{\mathrm{der}} & \longrightarrow & \mathbf{M} & \longrightarrow & GL_1{}^{k-1} & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{\mathbf{M}}_{\mathrm{der}} & \longrightarrow & \widetilde{\mathbf{M}} & \longrightarrow & GL_1{}^{k} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & & \longrightarrow & & \widetilde{\mathbf{M}}/\mathbf{M} & \xrightarrow{\simeq} & GL_1 & \longrightarrow & 1 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 1 & & 1 & &
\end{array}
$$

The maps $\mathbf{M} \to GL_1{}^{k-1}$ and $\widetilde{\mathbf{M}} \to GL_1{}^{k}$ are

$$(g_1, g_2, \cdots, g_{k-1}, g) \mapsto (\det g_1, \det g_2, \cdots, \det g_{k-1})$$

and

$$(g_1, g_2, \cdots, g_k) \mapsto (\det g_1, \det g_2, \cdots, \det g_k),$$

respectively, with det the determinant map. Further, the map $GL_1{}^k \to GL_1$ is the product

$$(a_1, a_2, \cdots, a_k) \mapsto a_1 \cdot a_2 \cdots a_k$$

so that the composite $\widetilde{\mathbf{M}} \to GL_1^k \to GL_1$ becomes the product of determinants. We then obtain an exact sequence (the middle vertical one)

$$(2.3) \qquad 1 \longrightarrow \mathbf{M} \longrightarrow \widetilde{\mathbf{M}} \longrightarrow GL_1 \longrightarrow 1$$

which yields

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow Z(\widehat{\widetilde{M}}) \simeq (\mathbb{C}^{\times})^k \longrightarrow Z(\widehat{M}) \longrightarrow 1$$

(cf. [Kot84, (1.8.1) p.616]). We note that the injective map $\mathbb{C}^{\times} \hookrightarrow (\mathbb{C}^{\times})^k$ is a diagonal embedding. Thus the center $Z(\widehat{M}) \simeq (\mathbb{C}^{\times})^k/\mathbb{C}^{\times}$ is connected. As another point of view, it follows from [Kot84, (1.8.3) p.616] that the center $Z(\widehat{M})$ is connected since $\mathbf{M}_{\mathrm{der}} = \prod_{i=1}^{k} SL_{n_i}$ is simply connected.

Remark 2.4. We also have the following commutative diagram of L-groups:

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{C}^{\times} & \overset{\simeq}{\longrightarrow} & \ker & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (\mathbb{C}^{\times})^k & \longrightarrow & \widehat{\widetilde{M}} & \longrightarrow & \widehat{(\widetilde{M}_{\mathrm{der}})} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & (\mathbb{C}^{\times})^k/\mathbb{C}^{\times} & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{(M_{\mathrm{der}})} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1
\end{array}
$$

So, we have an exact sequence (the middle vertical one)

$$ 1 \longrightarrow \mathbb{C}^{\times} \longrightarrow \widehat{\widetilde{M}} \longrightarrow \widehat{M} \longrightarrow 1. $$

Moreover, from (2.3) we see

$$ \mathbb{C}^{\times} = \widehat{GL_1} = \widehat{(\widetilde{M}/M)}. $$

Hence, we have

(2.4) $$ 1 \longrightarrow \widehat{(\widetilde{M}/M)} \longrightarrow \widehat{\widetilde{M}} \longrightarrow \widehat{M} \longrightarrow 1 $$

is also exact.

Remark 2.5. All arguments in Remarks 2.3 and 2.4 hold for the F-inner form $\mathbf{M}'$ of $\mathbf{M}$ as well. Further, we have a group isomorphism

(2.5) $$ \widetilde{M}/M \simeq F^{\times} \simeq \widetilde{M}'/M'. $$

Indeed, applying Galois cohomology to (2.3), we have

$$ 1 \longrightarrow \mathbf{M}(F) \longrightarrow \widetilde{\mathbf{M}}(F) \longrightarrow F^{\times} \longrightarrow H^1(F,\mathbf{M}) \longrightarrow H^1(F,\widetilde{\mathbf{M}}) \longrightarrow 1. $$

Note that $H^1(F,\widetilde{\mathbf{M}}) = 1$ from [PR94, Lemma 2.2]. Also, it is well known (cf. [Kot97, p.270]) that $H^1(F,\mathbf{M}) \hookrightarrow H^1(F,\mathbf{G})$ (true for any connected reductive F-group $\mathbf{G}$ and its F-Levi subgroup $\mathbf{M}$). Since $\mathbf{G} = SL_n$ is simply connected semi-simple, we have $H^1(F,\mathbf{M}) = H^1(F,\mathbf{G}) = 1$ due to [PR94, Theorem 6.4]. Therefore, we have $\widetilde{M}/M \simeq F^{\times}$. This argument is also true for M' since $H^1(F,\widetilde{\mathbf{M}}') = 1$ ([PR94, Lemma 2.8]) and $\mathbf{G}'$ is simply connected semi-simple as well. Thus, we have the isomorphism (2.5).

2.5. **Local Jacquet-Langlands correspondence.** Let $\mathbf{G}$ be GL_n over F and let $\mathbf{G}'$ be an F-inner form of $\mathbf{G}$. We denote by $\mathcal{E}^2(G)$ and $\mathcal{E}^2(G')$ the sets of essentially square-integrable representations in $\mathrm{Irr}(G)$ and $\mathrm{Irr}(G')$, respectively. We say a semisimple element $g \in G$ is *regular* if its characteristic polynomial has distinct roots in $\bar{F}$. We write G^{reg} for the set of regular semisimple elements in G. We denote by $C_c^{\infty}(G)$ the Hecke algebra of locally constant functions on G with compact support. Fix a Haar measure dg on G. For any $\rho \in \mathrm{Irr}(G)$, there is a unique locally constant function Θ_{ρ} on G^{reg} which is invariant under conjugation by G such that

$$ \mathrm{tr}\, \rho(f) = \int_{G^{\mathrm{reg}}} \Theta_{\rho}(g) f(g) dg, $$

for all $f \in C_c^{\infty}(G)$. Here

$$\rho(f) \cdot v = \int_G f(x)\rho(x)v \, dx.$$

In positive characteristic, it is required that the support of the function $f \in C_c^{\infty}(G)$ is contained in G^{reg}. We refer the reader to [HC81, p.96] and [DKV84, b. p.33] for details. The same is true for G'.

We say $g \in G^{reg}$ and $g' \in G'^{reg}$ correspond and denote this by $g \leftrightarrow g'$ if their characteristic polynomials are equal. We state the local Jacquet-Langlands correspondence ([DKV84, B.2.a], [Rog83, Theorem 5.8], and [Bad08, Theorem 2.2]) as follows.

Proposition 2.6. *There is a unique bijection* $\mathbf{C} : \mathcal{E}^2(G) \longrightarrow \mathcal{E}^2(G')$ *such that: for all* $\sigma \in \mathcal{E}^2(G)$, *we have*

$$\Theta_\sigma(g) = (-1)^{n-m}\Theta_{\mathbf{C}(\sigma)}(g')$$

whenever $g \leftrightarrow g'$.

Remark 2.7. For any $\sigma \in \mathcal{E}^2(G)$ and character η of $F^{\times}$, we have $\mathbf{C}(\sigma \otimes (\eta \circ \det)) = \mathbf{C}(\sigma) \otimes (\eta \circ \mathrm{Nrd})$ due to [DKV84, Introduction d.4)]. Here Nrd is the reduced norm on G'. It is known that any character on G (respectively, G') is of the form $\eta \circ \det$ (respectively, $\eta \circ \mathrm{Nrd}$) for some character η on $F^{\times}$. The local Jacquet-Langlands correspondence can be generalized to the case that G is a product of a general linear groups in an obvious way.

2.6. **Restriction of representations.** We recall the results of Tadić in [Tad92]. Throughout Section 2.6, $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ denote connected reductive groups over F, such that

$$(2.6) \qquad\qquad \mathbf{G}_{\mathrm{der}} = \widetilde{\mathbf{G}}_{\mathrm{der}} \subseteq \mathbf{G} \subseteq \widetilde{\mathbf{G}},$$

where $\mathbf{G}_{\mathrm{der}}$ and $\widetilde{\mathbf{G}}_{\mathrm{der}}$ denote the derived groups of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$, respectively.

Proposition 2.8. *(*[Tad92, Lemma 2.1 and Proposition 2.2]*) For any* $\sigma \in \mathrm{Irr}(G)$, *there exists* $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ *such that* $\sigma \hookrightarrow \widetilde{\sigma}|_G$, *that is,* σ *is isomorphic to an irreducible constituent of the restriction* $\widetilde{\sigma}|_G$ *of* $\widetilde{\sigma}$ *to* G.

Given $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ as in Proposition 2.8, we denote by $\Pi_{\widetilde{\sigma}}(G)$ the set of equivalence classes of all irreducible constituents of $\widetilde{\sigma}|_G$.

Remark 2.9. ([Tad92, Proposition 2.7]) Any member in $\Pi_{\widetilde{\sigma}}(G)$ is supercuspidal, essentially square-integrable, discrete series or tempered if and only if $\widetilde{\sigma}$ is.

Proposition 2.10. *(*[Tad92, Corollary 2.5]*) Let* $\widetilde{\sigma}_1$ *and* $\widetilde{\sigma}_2 \in \mathrm{Irr}(\widetilde{G})$ *be given. Then the following statements are equivalent:*

(i) *There exists a character* $\eta \in (\widetilde{G}/G)^D$ *such that* $\widetilde{\sigma}_1 \simeq \widetilde{\sigma}_2 \otimes \eta$;
(ii) $\Pi_{\widetilde{\sigma}_1}(G) \cap \Pi_{\widetilde{\sigma}_2}(G) \neq \emptyset$;
(iii) $\Pi_{\widetilde{\sigma}_1}(G) = \Pi_{\widetilde{\sigma}_2}(G)$.

Let $\sigma \in \Pi_{\mathrm{disc}}(G)$ be given. Choose $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ as in Proposition 2.8. For any $\tilde{g} \in \widetilde{G}$, we define an action $\tilde{x} \mapsto {}^{\tilde{g}}\widetilde{\sigma}(\tilde{x}) := \widetilde{\sigma}(\tilde{g}^{-1}\tilde{x}\tilde{g})$ on the space of $\widetilde{\sigma}$. Since G is a normal subgroup of $\widetilde{G}$, the restriction of the action of $\widetilde{G}$ on $\widetilde{\sigma}$ to G induces the action $x \mapsto {}^{\tilde{g}}\sigma(x) := \sigma(\tilde{g}^{-1}x\tilde{g})$ on the space of σ. It is clear that ${}^{\tilde{g}}\widetilde{\sigma} \simeq \widetilde{\sigma}$ as representations of $\widetilde{G}$. Hence, it turns out that $\widetilde{G}$ acts on the set $\Pi_{\widetilde{\sigma}}(G)$ by conjugation.

Lemma 2.11. *The group $\widetilde{G}$ acts transitively on the set $\Pi_{\widetilde{\sigma}}(G)$.*

Proof. Let σ_1 and σ_2 be given in $\Pi_{\widetilde{\sigma}}(G)$. Denote by V_i the space of σ_i for $i = 1, 2$. Since $\widetilde{\sigma}$ is irreducible, there exists $\tilde{g} \in \widetilde{G}$ such that $\widetilde{\sigma}(\tilde{g})V_1 = V_2$. So, we have $\widetilde{\sigma}(\tilde{g})\sigma_1\widetilde{\sigma}(\tilde{g}^{-1}) \simeq \sigma_2$. Hence, we have $\tilde{g} \in \widetilde{G}$ such that $^{\tilde{g}}\sigma_1 \simeq \sigma_2$. $\qquad\square$

We define the stabilizer of σ in $\widetilde{G}$

$$\widetilde{G}_\sigma := \{\tilde{g} \in \widetilde{G} : {}^{\tilde{g}}\sigma \simeq \sigma\}.$$

It is known [Tad92, Corollary 2.3] that $\widetilde{G}_\sigma$ is an open normal subgroup of $\widetilde{G}$ of finite index and satisfies

$$Z(\widetilde{G}) \cdot G \subseteq \widetilde{G}_\sigma \subseteq \widetilde{G}.$$

Hence, we have proved the following proposition (cf. [GK82, Lemma 2.1]).

Proposition 2.12. *Let $\sigma \in \Pi_{\mathrm{disc}}(G)$ be given. Choose $\widetilde{\sigma} \in \mathrm{Irr}(G)$ as in Proposition 2.8. The quotient of $\widetilde{G}/\widetilde{G}_\sigma$ acts by conjugation on the set $\Pi_{\widetilde{\sigma}}(G)$ simply and transitively. In fact, there is a bijection between $\widetilde{G}/\widetilde{G}_\sigma$ and $\Pi_{\widetilde{\sigma}}(G)$.*

Note that the index $[\widetilde{G} : (Z(\widetilde{G}) \cdot G)]$ is also finite since $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ share the same derived group by the condition (2.6) (cf. Section 3.2).

Remark 2.13. Due to due to Propositions 2.10 and 2.12, the set $\Pi_{\widetilde{\sigma}}(G)$ is finite and independent of the choice of $\widetilde{\sigma}$.

3. Elliptic tempered L-packets for Levi subgroups

In this section we construct elliptic tempered L-packets for F-Levi subgroups of SL_n and its F-inner form. Our main tools are the local Langlands correspondence for GL_n in [HT01, Hen00], the local Jacquet-Langlands correspondence in Section 2.5 and the result of Labesse in [Lab85]. Further, we discuss the size of our L-packets in terms of Galois cohomology (Proposition 3.5).

Throughout Section 3, we continue with the notation in Section 2. Let $\mathbf{G}$ and $\mathbf{G}'$ be SL_n and its F-inner form. Let $\mathbf{M}$ and $\mathbf{M}'$ be F-Levi subgroups of $\mathbf{G}$ and $\mathbf{G}'$ as in (2.1) and (2.2), respectively. We identify $Z(\mathbf{M})$ and $Z(\mathbf{M}')$. Note that $\mathbf{M}$ is split over F, so we take $^L M = \widehat{M}$. Similarly, we take $^L \widetilde{M} = \widehat{\widetilde{M}} = \prod_{i=1}^k GL_{n_i}(\mathbb{C})$ (cf. (2.4)). Note that $Z(\widehat{M})^{\Gamma_F} = Z(\widehat{M})$ and $Z(\widehat{\widetilde{M}})^{\Gamma_F} = Z(\widehat{\widetilde{M}})$. Here, the superscript Γ_F means the group of Γ_F-invariants. We identify $\widehat{M}$ and $\widehat{M}'$.

3.1. **Construction of L-packets.** Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ be an L-parameter. We denote by $C_\phi(\widehat{M})$ the centralizer of the image of ϕ in $\widehat{M}$. We note that $Z(\widehat{M}) \subseteq C_\phi(\widehat{M})$. We denote by $S_\phi(\widehat{M})$ the group $\pi_0(C_\phi(\widehat{M})) := C_\phi(\widehat{M})/C_\phi(\widehat{M})^\circ$ of connected components.

Definition 3.1. We say that $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ is *elliptic* if the quotient group $C_\phi(\widehat{M})/Z(\widehat{M})$ is finite; and ϕ is *tempered* if $\phi(W_F)$ is bounded.

Remark 3.2. The parameter ϕ is elliptic if and only if the image of ϕ in $\widehat{M}$ is not contained in any proper Levi subgroup of $\widehat{M}$. Moreover, this is equivalent to requiring that the connected component $C_\phi(\widehat{M})^\circ$ is contained in $Z(\widehat{M})$ (cf. [Kot84, (10.3)]).

The rest of Section 3.1 is devoted to a construction of L-packets on M and M' associated to an elliptic tempered L-parameter. Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ be an elliptic tempered L-parameter. Recall the exact sequence (2.4)

$$1 \longrightarrow (\widehat{\widetilde{M}/M}) \simeq \mathbb{C}^\times \longrightarrow \widehat{\widetilde{M}} \xrightarrow{pr} \widehat{M} \longrightarrow 1.$$

By [Lab85, Théorèm 8.1], we have an (elliptic tempered) L-parameter

$$\widetilde{\phi} : W_F \times SL_2(\mathbb{C}) \to \widehat{\widetilde{M}}$$

of $\widetilde{M}$ such that $pr \circ \widetilde{\phi} = \phi$ (see [Wei74] and [Hen80] for the case $M = SL_n$ and $\widetilde{M} = GL_n$).

Remark 3.3. Such a parameter $\widetilde{\phi}$ is determined up to a 1-cocycle of W_F in $(\widehat{\widetilde{M}/M})$ (see [Lab85, Section 7] where $(\widehat{\widetilde{M}/M})$ is a central torus $^L S^\circ$).

By the local Langlands correspondence for GL_n [HT01, Hen00], we have a unique discrete series representation $\widetilde{\sigma} \in \Pi_{\mathrm{disc}}(\widetilde{M})$ associated to the L-parameter $\widetilde{\phi}$. We define an L-packet

$$\Pi_\phi(M) := \{\tau : \tau \hookrightarrow \widetilde{\sigma}|_M\}.$$

Note that, if $\widetilde{\phi}_1$ is another lift of ϕ, then we have $\widetilde{\phi}_1 \simeq \widetilde{\phi} \otimes \chi$, for some 1-cocycle χ of W_F in $(\widehat{\widetilde{M}/M})$, due to Remark 3.3. Hence, by the local Langlands correspondence for GL_n again, $\widetilde{\phi}_1$ gives another discrete series representation $\widetilde{\sigma} \otimes \chi \in \Pi_{\mathrm{disc}}(\widetilde{M})$. Here χ denotes the character on $\widetilde{M}/M$ associated to the 1-cocycle χ (by abuse of notation). It is then clear $(\widetilde{\sigma} \otimes \chi)|_M = \widetilde{\sigma}|_M$. Hence, the representation $\widetilde{\sigma} \otimes \chi$ associated to $\widetilde{\phi}_1$ via the local Langlands correspondence gives the same L-packet $\Pi_\phi(M)$ on M.

On the other hand, given $\sigma \in \Pi_{\mathrm{disc}}(M)$, there exists a lift $\widetilde{\sigma} \in \Pi_{\mathrm{disc}}(\widetilde{M})$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_M$ (see Section 2.6). By the local Langlands correspondence for GL_n [HT01, Hen00], we have a unique elliptic tempered L-parameter $\widetilde{\phi} : W_F \times SL_2(\mathbb{C}) \to \widehat{\widetilde{M}}$ corresponding to $\widetilde{\sigma}$. Hence, we obtain an elliptic tempered L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ by composing with the projection $\widehat{\widetilde{M}} \twoheadrightarrow \widehat{M}$. If we choose another lift $\widetilde{\sigma}_1 \in \Pi_{\mathrm{disc}}(\widetilde{M})$ of σ, then we have $\widetilde{\sigma}_1 \simeq \widetilde{\sigma} \otimes \chi$ for some character χ on $\widetilde{M}$ such that χ is trivial on M (see Section 2.6). So, the local Langlands correspondence for GL_n yields the L-parameter $\widetilde{\phi} \otimes \chi$ corresponding to $\widetilde{\sigma}_1$. But, considering the projection $\mathrm{pr} : \widehat{\widetilde{M}} \twoheadrightarrow \widehat{M}$, the composite $\mathrm{pr} \circ (\widetilde{\phi} \otimes \chi)$ has to be identical to ϕ since $\mathrm{pr} \circ \chi$ vanishes on $\widehat{M}$. Hence we have verified that, given $\sigma \in \Pi_{\mathrm{disc}}(M)$, we have a unique elliptic tempered L-parameter ϕ corresponding to σ. Therefore, the L-packets of the form $\Pi_\phi(M)$ exhaust all irreducible discrete series representations of M.

Consider ϕ and $\widetilde{\phi}$ as L-parameters of M' and $\widetilde{M}'$, respectively. The local Jacquet-Langlands correspondence gives a unique discrete series representation $\widetilde{\sigma}' \in \Pi_{\mathrm{disc}}(\widetilde{M}')$ which corresponds to the above $\widetilde{\sigma} \in \Pi_{\mathrm{disc}}(\widetilde{M})$. We define an L-packet

$$\Pi_\phi(M') := \{\tau' : \tau' \hookrightarrow \widetilde{\sigma}'|_{M'}\}$$

(cf. [HS11, Chapter 11]). In the same way with the split case M, the L-packets of the form $\Pi_\phi(M')$ exhaust all irreducible discrete series representations of M' (cf. [GK82, Section 4] and [HS11, Chapter 12]).

Through a natural embedding $\widehat{M} \hookrightarrow \widehat{G} = PGL_n(\mathbb{C})$, we define an L-packet $\Pi_\phi(G)$ of G associated to the L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{G}$ as

$$\Pi_\phi(G) := \{\text{all irreducible constituents of } i_{GM}(\tau) : \tau \in \Pi_\phi(M)\}.$$

In a similar manner we define an L-packet $\Pi_\phi(G')$ of G'. Recall that $\widehat{G} = \widehat{G'}$. We let $C_\phi(\widehat{G})$ be the centralizer of the image of ϕ in $\widehat{G}$. We denote by $S_\phi(\widehat{G})$ the group $\pi_0(C_\phi(\widehat{G})) := C_\phi(\widehat{G})/C_\phi(\widehat{G})^\circ$ of connected components.

3.2. **Sizes of L-packets.** We recall from Section 2.6 that, for any $\sigma \in \Pi_{\mathrm{disc}}(M)$, $\widetilde{M}$ acts on the set $\Pi_{\widetilde{\sigma}}(M)$ by conjugation and the action is transitive. Note that $\Pi_{\widetilde{\sigma}}(M) = \Pi_\phi(M)$, where ϕ is the L-parameter associated to σ.

Lemma 3.4. *Let $\Pi_\phi(M)$ be an L-packet associated to an elliptic tempered L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$. Then the quotient of $\widetilde{M}/\widetilde{M}_\sigma$ acts by conjugation on $\Pi_\phi(M)$ simply and transitively. In fact, there is a bijection between $\widetilde{M}/\widetilde{M}_\sigma$ and $\Pi_\phi(M)$. The same is true for an L-packet $\Pi_\phi(M')$ of M'.*

Proof. This is a consequence of Proposition 2.12 and our construction of L-packets in Section 3.1. $\qquad\square$

We now consider the cardinality of the L-packets $\Pi_\phi(M)$ and $\Pi_\phi(M')$.

Proposition 3.5. *Let $\Pi_\phi(M)$ be an L-packet associated to an an elliptic tempered L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$. Then we have*

$$|\Pi_\phi(M')| \ \Big| \ |\Pi_\phi(M)| \ \text{ and } \ |\Pi_\phi(M)| \ \Big| \ \left|H^1(F, \pi_0(Z(\mathbf{M})))\right|.$$

Proof. We denote by $\mathbf{m}(\sigma)$ and $\mathbf{m}(\sigma')$ the multiplicities of σ and σ' in $\widetilde{\sigma}|_M$ and $\widetilde{\sigma}'|_{M'}$, respectively. We set

$$Y(\widetilde{\sigma}) := \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta\},$$
$$Y(\widetilde{\sigma}') := \{\eta \in (\widetilde{M'}/M')^D : \widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta\}.$$

From Remarks 2.5 and 2.7 we see $Y(\widetilde{\sigma}') = Y(\widetilde{\sigma})$. Then, from [Tad92, Proposition 2.4], we have

$$|Y(\widetilde{\sigma}')| = \dim_{\mathbb{C}} \mathrm{Hom}_{M'}(\widetilde{\sigma}', \widetilde{\sigma}').$$

We note that $\dim_{\mathbb{C}} \mathrm{Hom}_{M'}(\widetilde{\sigma}', \widetilde{\sigma}')$ equals the product of $\mathbf{m}(\sigma')^2$ and the number of irreducible inequivalent constituents in $\widetilde{\sigma}'|_{M'}$ by Schur's lemma. Since $\Pi_{\widetilde{\sigma}'}(M')$ is the set of irreducible inequivalent constituents in $\widetilde{\sigma}'|_{M'}$ by definition, we thus have

$$(3.1) \qquad\qquad |Y(\widetilde{\sigma}')| = |\Pi_{\widetilde{\sigma}'}(M')| \cdot \mathbf{m}(\sigma')^2.$$

Note (3.1) holds for M, as well. Since the multiplicity $\mathbf{m}(\sigma)$ of σ in $\widetilde{\sigma}|_M$ equals 1, by [Tad92, Proposition 2.8], we have $|Y(\widetilde{\sigma})| = |\Pi_{\widetilde{\sigma}}(M)|$. Hence, we have proved the first assertion $|\Pi_\phi(M')| \ \Big| \ |\Pi_\phi(M)|$.

Consider a homomorphism $\lambda : Z(\mathbf{M}) \times \mathbf{M} \to \widetilde{\mathbf{M}}$ defined as $\lambda(z, m) = zm$. Since $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ have the same derived group, we get an exact sequence of algebraic groups

$$(3.2) \qquad\qquad 1 \to Z(\mathbf{M}) \to Z(\widetilde{\mathbf{M}}) \times \mathbf{M} \xrightarrow{\lambda} \widetilde{\mathbf{M}} \to 1.$$

We note that the injection of $Z(\mathbf{M})$ into $Z(\widetilde{\mathbf{M}}) \times \mathbf{M}$ is $z \mapsto (z, z^{-1})$. Applying Galois cohomology to (3.2), we have an exact sequence

$$\cdots \to Z(\widetilde{M}) \times M \xrightarrow{\lambda} \widetilde{M} \to H^1(F, Z(\mathbf{M})) \to H^1(F, Z(\widetilde{\mathbf{M}}) \times \mathbf{M}) \to \cdots.$$

Note that $H^1(F, Z(\widetilde{\mathbf{M}}) \times \mathbf{M}) = H^1(F, Z(\widetilde{\mathbf{M}})) \times H^1(F, \mathbf{M}) = 1$ (see Remark 2.5) and the cokernel of λ is $\widetilde{M}/(Z(\widetilde{M}) \cdot M)$. Consider another exact sequence

$$1 \to Z(\mathbf{M})^\circ \to Z(\mathbf{M}) \to \pi_0(Z(\mathbf{M})) \to 1.$$

Since $H^1(F, Z(\mathbf{M})^\circ) = 1$ by Hilbert's Theorem 90, we have

$$H^1(F, Z(\mathbf{M})) \hookrightarrow H^1(F, \pi_0(Z(\mathbf{M}))).$$

Hence, $\widetilde{M}/(Z(\widetilde{M}) \cdot M)$ must be a subgroup of $H^1(F, \pi_0(Z(\mathbf{M})))$. So, we have

$$\widetilde{M}/(Z(\widetilde{M}) \cdot M) \hookrightarrow H^1(F, Z(\mathbf{M})) \hookrightarrow H^1(F, \pi_0(Z(\mathbf{M}))).$$

Since $(Z(\widetilde{M}) \cdot M) \subseteq \widetilde{M}_\sigma \subseteq \widetilde{M}$ [Tad92, Corollary 2.3], we deduce the second assertion from Lemma 3.4. Thus, the proof is complete. $\square$

Example 3.6. Suppose $\mathbf{G} = GL_{n_1+n_2}$, and $\mathbf{M} = GL_{n_1} \times GL_{n_2}$. Since $Z(\mathbf{M})$ is connected, any L-packet of M is a singleton which has been proved in [HT01, Hen00].

Example 3.7. Suppose $F = \mathbb{Q}_p$, $p \neq 2$, $M = G = SL_2(F)$ and $M' = G' = SL_1(D)$, where D is the quaternion division algebra over F. Note that $H^1(F, \pi_0(Z(\mathbf{M}))) \simeq F^\times/(F^\times)^2$, since $Z(\mathbf{M}) = Z(\mathbf{M}') \simeq \mu_2 = \pi_0(Z(\mathbf{M}))$. It turns out that

$$\left| H^1(F, \pi_0(Z(\mathbf{M}))) \right| = 4.$$

Hence, by Proposition 3.5, the cardinality of any an elliptic tempered L-packet of M is either 1, 2 or 4. Also, the cardinality of an elliptic tempered L-packets of M' can be determined using the first assertion of Proposition 3.5.

Corollary 3.8. *If $\pi_0(Z(\mathbf{M})) = 1$, that is, $Z(\mathbf{M})$ is connected, then every an elliptic tempered L-packet of M is a singleton.*

Example 3.9. Suppose $F = \mathbb{Q}_p$, $\mathbf{G} = SL_3$ and $\mathbf{M} = (GL_2 \times GL_1) \cap SL_3$. Since the coordinate ring $F[Z(\mathbf{M})] = F[x, y]/(x^2y - 1)$ and $x^2y - 1$ is irreducible, $Z(\mathbf{M})$ is connected. Hence, due to Corollary 3.8, any an elliptic tempered L-packet of M is singleton. This argument is clear for the obvious reason that $\mathbf{M} \simeq GL_2$.

Remark 3.10. Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{G}$ be a tempered L-parameter. Let $\widehat{M}$ be a minimal Levi subgroup in the sense that $\widehat{M}$ contains the image of ϕ (see [Bor79, Section 3.4] for the definition of Levi subgroup of $^L M$). Then ϕ becomes an elliptic tempered parameter of M. Choose a member $\tau \in \Pi_\phi(M)$ and recall the Knapp-Stein R-group R_τ for τ. Then, from Lemma 3.4 we have

$$|\Pi_\phi(G)| \ \Big| \ \big(|R_\tau| \cdot |\Pi_\phi(M)| \big) \ \Big| \ \big(|R_\tau| \cdot \left| H^1(F, \pi_0(Z(\mathbf{M}))) \right| \big).$$

The same is true for $\mathbf{G}'$. In fact, for the split case $\mathbf{G}$, we have an equality

$$|\Pi_\phi(G)| = |R_\tau| \cdot |\Pi_\phi(M)|$$

from Theorem 4.2 and Proposition 4.3 in Section 4.1.

Remark 3.11. All statements in Section 3 admit an obvious generalization to the case of any connected reductive group $\mathbf{M}$ over F such that $\mathbf{M}_{\mathrm{der}} = \widetilde{\mathbf{M}}_{\mathrm{der}} \subseteq \mathbf{M} \subseteq \widetilde{\mathbf{M}}$, where $\widetilde{\mathbf{M}} = \prod_{i=1}^{k} GL_{n_i}$.

4. R-GROUPS FOR SL_n AND ITS INNER FORMS

In this section we first review the results of the second named author [Gol94a] and Gelbart-Knapp [GK82] and prove that Knapp-Stein, Arthur and Endoscopic R-groups are all identical for SL_n (Theorem 4.8). Second, we discuss the Knapp-Stein R-group for an F-inner form of SL_n and establish its connection with R-groups for SL_n. Throughout Section 4, we continue with the notation in Sections 2 and 3.

4.1. R-groups for SL_n Revisited. Let $\widetilde{\mathbf{G}} = GL_n$ and $\mathbf{G} = SL_n$ be over F. Let $\mathbf{M}$ be an F-Levi subgroup of $\mathbf{G}$, and let $\widetilde{\mathbf{M}}$ be an F-Levi subgroup of $\widetilde{\mathbf{G}}$ such that $\widetilde{\mathbf{M}} = \mathbf{M} \cap \mathbf{G}$. Let $\sigma \in \Pi_{\mathrm{disc}}(M)$ be given. Choose $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{M})$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_M$. We identify Weyl groups W_M and $W_{\widetilde{M}}$ as a subgroup of the group S_k of permutations on k letters. Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$ be an elliptic tempered L-parameter associated to σ. Let $\Pi_\phi(M)$ be an L-packet of M associated to ϕ. We define the following groups:

$$\bar{L}(\widetilde{\sigma}) := \{\eta \in (\widetilde{M}/M)^D : {}^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta \text{ for some } w \in W_M\},$$

$$X(\widetilde{\sigma}) := \{\eta \in (\widetilde{M}/M)^D : \widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta\},$$

$$X(i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma})) := \{\eta \in (\widetilde{G}/G)^D : i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \simeq i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \otimes \eta\}.$$

Since any character of $GL_n(F)$ is of the form $\eta \circ \det$ for some character of $F^\times$, we often make no distinct between η and $\eta \circ \det$. Further, since $\widetilde{\mathbf{M}}$ is of the form $\prod_{i=1}^{k} GL_{n_i}$, we simply write η for $\prod_{i=1}^{k} (\eta_i \circ \det_i) \in (\widetilde{M}/M)^D$, where η_i denotes a character of $F^\times$ and $\det_i$ denotes the determinant of $GL_{n_i}(F)$ for each i.

Lemma 4.1. *(Goldberg, [Gol94a, Lemma 2.3]) Let $\sigma \in \Pi_{\mathrm{disc}}(M)$ be given. For any lift $\widetilde{\sigma} \in \Pi_{\mathrm{disc}}(\widetilde{M})$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_M$, we have*

$$W(\sigma) = \{w \in W_M : {}^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta \text{ for some } \eta \in (\widetilde{M}/M)^D\}.$$

Theorem 4.2. *(Goldberg, [Gol94a, Theorem 2.4]) The Knapp-Stein R-group R_σ is isomorphic to $\bar{L}(\widetilde{\sigma})/X(\widetilde{\sigma})$.*

Note that $i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma})$ is always irreducible since $\widetilde{\sigma}$ is a discrete series representation [BZ77, Theorem 4.2].

Proposition 4.3. *(Gelbart-Knapp, [GK82, Theorem 4.3]) There are group isomorphisms*

$$X(\widetilde{\sigma}) \simeq S_\phi(\widehat{M}) \text{ and } X(i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma})) \simeq S_\phi(\widehat{G}).$$

Consequently, both $S_\phi(\widehat{M})$ and $S_\phi(\widehat{G})$ are finite abelian groups. Further, $S_\phi(\widehat{M})^D$ and $S_\phi(\widehat{G})^D$ have canonical simply transitive group actions on $\Pi_\phi(M)$ and $\Pi_\phi(G)$, respectively.

Remark 4.4. Since $X(\widetilde{\sigma})$ is a subgroup of $X(i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}))$, we notice that there is an embedding $\iota : S_\phi(\widehat{M}) \hookrightarrow S_\phi(\widehat{G})$. We abuse notation to denote the quotient $S_\phi(\widehat{G})/\iota(S_\phi(\widehat{M}))$ by $S_\phi(\widehat{G})/S_\phi(\widehat{M})$. We call the quotient $S_\phi(\widehat{G})/S_\phi(\widehat{M})$ the *Endoscopic R-group*. We refer the reader to [Art89, (7.1)] where the Endoscopic R-group is denoted by R_ϕ.

We have the following proposition.

Proposition 4.5. *We have the identity*

$$\bar{L}(\widetilde{\sigma}) = X(i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}))$$

as subgroups of the group $(F^\times)^D$ of characters on $F^\times$.

Proof. Let $\eta \in (\widetilde{G}/G)^D$ be given. We first note that $i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \otimes \eta \simeq i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma} \otimes \eta)$. From the Langlands classification [Kud94, Theorem 1.2.5 (b)], we have

$$i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \simeq i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma} \otimes \eta)$$

if and only if

$$^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta \text{ for some } w \in W_M.$$

Now, the proposition follows from the definition of $\bar{L}(\widetilde{\sigma})$. $\square$

Theorem 4.6. *The Knapp-Stein R-group R_σ is isomorphic to $S_\phi(\widehat{G})/S_\phi(\widehat{M})$.*

Proof. This is a consequence of Theorem 4.2 and Propositions 4.3 and 4.5 (cf. [Sha83, Proposition 9.1] for another proof). $\square$

In what follows, we give a connection between the Knapp-Stein R-group R_σ and the Arthur R-group $R_{\phi,\sigma}$ for the case when σ is in $\Pi_{\mathrm{disc}}(M)$ and $\mathbf{M}$ is an F-Levi subgroup of $\mathbf{G} = SL_n$.

Lemma 4.7. *We have the equalities*

$$W_\phi = W_{\phi,\sigma} \text{ and } W_\phi^\circ = W_{\phi,\sigma}^\circ.$$

Proof. The former implies the latter. So, it is enough to show that $W_\phi \subset W_{\phi,\sigma}$. Let $w \in W_\phi$ be given. From [BJ04, Lemma 2.3], we note that the element w lies in $W_{\widehat{M}}$ satisfying $^w\phi \simeq \phi$, i.e., $^w\phi$ is conjugate to ϕ in $\widehat{M}$ (since $w \in C_\phi(\widehat{M})$). It follows that $^w\tilde{\phi} \simeq \tilde{\phi} \otimes \eta$ for some $\eta \in (F^\times)^D$. We identify W_M and $W_{\widehat{M}}$. From the local Langlands correspondence for GL_n and the W_M-action on the L-parameter $\tilde{\phi} = \tilde{\phi}_1 \oplus \cdots \oplus \tilde{\phi}_k$, we have

$$^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta$$

for some $\eta \in (\widetilde{M}/M)^D$. Now, from Lemma 4.1 we see w must be in $W(\sigma)$. By the definition $W_{\phi,\sigma} = W_\phi \cap W(\sigma)$, which completes the proof. $\square$

Hence, from [Art89, (7.1) & p.44], we get

(4.1) $R_\phi = R_{\phi,\sigma} \simeq S_\phi(\widehat{G})/S_\phi(\widehat{M}).$

Thus, due to Theorems 4.2 and 4.6, we have proved the following.

Theorem 4.8. *Let $\mathbf{M}$ be an F-Levi subgroup of $\mathbf{G} = SL_n$. Let $\Pi_\phi(M)$ be an L-packet on M associated with an elliptic tempered L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M}$. For any $\sigma \in \Pi_\phi(M)$, we have*

$$R_\phi = R_{\phi,\sigma} \simeq S_\phi(\widehat{G})/S_\phi(\widehat{M}) \simeq \bar{L}(\widetilde{\sigma})/X(\widetilde{\sigma}) \simeq R_\sigma.$$

4.2. **R-groups for F-inner Forms of SL_n.** Let $\mathbf{M}'$ be an F-Levi subgroup of an inner form $\mathbf{G}'$ of $\mathbf{G} = SL_n$. Let $\sigma' \in \Pi_{\text{disc}}(M')$ be given. Choose $\widetilde{\sigma}' \in \Pi_{\text{disc}}(\widetilde{M}')$ such that $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$.

As in Section 2.4, we identify W_M, $W_{M'}$ and $W_{\widetilde{M}'}$. We also identify $\widetilde{M}/M$ and $\widetilde{M}'/M'$ (see Remark 2.5). In the same manner as in Section 4.1, we define

$$\bar{L}(\widetilde{\sigma}') := \{\eta' \in (\widetilde{M}'/M')^D : {}^w\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta' \text{ for some } w \in W_{M'}\},$$

$$X(\widetilde{\sigma}') := \{\eta' \in (\widetilde{M}'/M')^D : \widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta'\},$$

$$X(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}')) := \{\eta' \in (\widetilde{G}'/G')^D : i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}') \simeq i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}') \otimes \eta'\}.$$

Since any character of $GL_m(D)$ is of the form $\eta \circ \text{Nrd}$ for some character of $F^\times$ (cf. [BH06, Proposition 53.5]), we often make no distinct between η and $\eta \circ \text{Nrd}$. Further, since $\widetilde{M}'$ is of the form $\Pi_{i=1}^r GL_{m_i}(D)$, we simply write η for $\Pi_{i=1}^r(\eta_i \circ \text{Nrd}_i) \in (\widetilde{M}'/M')^D$, where η_i denotes a character of $F^\times$ and Nrd_i denotes the reduced norm of $GL_{m_i}(D)$ for each i.

Proposition 4.9. *Let $\sigma' \in \Pi_{\text{disc}}(M')$ be given. Choose $\widetilde{\sigma}' \in \Pi(\widetilde{M}')$ such that $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$. Then, we have*

$$W(\sigma') \subseteq \{w \in W_{M'} : {}^w\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta' \text{ for some } \eta' \in (\widetilde{M}'/M')^D\} = W(\sigma).$$

Proof. Let $w \in W(\sigma')$ be given. Then it turns out that ${}^w\widetilde{\sigma}'$ and $\widetilde{\sigma}'$ share an irreducible constituent ${}^w\sigma \simeq \sigma$. So, Proposition 2.10 implies that ${}^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta'$ for some character $\eta' \in (\widetilde{M}'/M')^D$. This proves the first inclusion.

For the second inclusion, we note that ${}^w\widetilde{\sigma}'$ and $\widetilde{\sigma}' \otimes \eta'$ correspond respectively to ${}^w\widetilde{\sigma}$ and $\widetilde{\sigma} \otimes \eta'$ by the local Jacquet-Langlands correspondence (see Section 2.5). So we have from Remarks 2.5 and 2.7

$${}^w\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta' \text{ for some } \eta' \in (\widetilde{M}'/M')^D$$

if and only if

$${}^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta \text{ for some } \eta \in (\widetilde{M}/M)^D.$$

Here we identify η and η' via the isomorphism $\widetilde{M}'/M' \simeq \widetilde{M}/M$ in Remark 2.5. Applying Lemma 4.1, the proof is complete. $\square$

From now on, we let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M} = \widehat{M}'$ be an elliptic tempered L-parameter. Let $\Pi_\phi(M)$ and $\Pi_\phi(M')$ be L-packets of M and M' associated to ϕ, respectively.

Remark 4.10. Let $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ be given. Lemma 4.1 and Proposition 4.9 provide the following diagram.

$$
\begin{array}{ccc}
{}^w\widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \eta & \overset{LJL}{\Longleftrightarrow} & {}^w\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta \\
\Updownarrow & & \Big\Updownarrow \\
{}^w\sigma \simeq \sigma & \overset{---\to}{\longleftarrow} & {}^w\sigma' \simeq \sigma'
\end{array}
$$

where w lies in $W_M = W_{M'} = W_{\widetilde{M}} = W_{\widetilde{M}'}$ and η is a character on $\widetilde{M}/M \simeq \widetilde{M}'/M'$ (see Remark 2.5).

Proposition 4.11. *For any $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$, we have*

$$W'_\sigma = W'_{\sigma'}$$

as subgroups of the group S_r of permutations on r letters.

Proof. We note that the sets $\Phi(P, A_M)$ and $\Phi(P', A_{M'})$ of reduced roots are identical from [Gol94a, p.85]. Let $\beta \in \Phi(P, A_M) = \Phi(P', A_{M'})$ be given. Since the Plancherel measures are invariant on $\Pi_\phi(M')$ due to [Sha91, Corollary 7.2], we have from [Cho12a, Theorem 6.3]

(4.2) $$\mu_\beta(\sigma) = \mu_\beta(\sigma').$$

In particular, $\mu_\beta(\sigma) = 0$ if and only if $\mu_\beta(\sigma') = 0$. From the definitions of W'_σ and $W'_{\sigma'}$ (see Section 2.2), the lemma follows. $\square$

Remark 4.12. Equation (4.2) can be also deduced from [Cho12a, Theorem 5.7] since our discrete series representations in $\Pi_\phi(M)$ and $\Pi_\phi(M')$ satisfy the character identity

$$\sum_{\sigma \in \Pi_\phi(M)} \Theta_\sigma(\gamma) = (-1)^{n-m} \sum_{\sigma' \in \Pi_\phi(M')} \mathbf{m}(\sigma')\Theta_{\sigma'}(\gamma')$$

for any γ and γ' have matching conjugacy classes. Here $\mathbf{m}(\sigma')$ is the multiplicity of σ' in $\widetilde{\sigma}'|_{M'}$ and $\Theta_\natural$ is the character function (Harish-Chandra character) of $\natural$. This character identity comes from the restriction of the character identity in the local Jacquet-Langlands correspondence between $\widetilde{M}$ and $\widetilde{M}'$ (Proposition 2.6). We also notice that the multiplicity $\mathbf{m}(\sigma) = 1$ [Tad92, Theorem 1.2]. Further, we refer the reader to [AC89, (2.5) on p.88] and [Sha91, Corollary 7.2] for another approach.

Now we state a relation between R-groups for $G = SL_n(F)$ and $G' = SL_m(D)$ as follows.

Theorem 4.13. *Let $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ be given. For any liftings $\widetilde{\sigma} \in \Pi_{\mathrm{disc}}(\widetilde{M})$ and $\widetilde{\sigma}' \in \Pi_{\mathrm{disc}}(\widetilde{M}')$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_{M'}$ and $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$, we have*

$$R_{\sigma'} \hookrightarrow \bar{L}(\widetilde{\sigma}')/X(\widetilde{\sigma}') \simeq R_\sigma \simeq R_{\phi,\sigma} = R_\phi.$$

Proof. From Propositions 4.9 and 4.11, it follows that $R_{\sigma'} \subseteq R_\sigma$. Since ${}^w\widetilde{\sigma}'$ and $\widetilde{\sigma}' \otimes \eta$ correspond respectively to ${}^w\widetilde{\sigma}$ and $\widetilde{\sigma} \otimes \eta$ by the local Jacquet-Langlands correspondence (see Section 2.5), we have

$$\bar{L}(\widetilde{\sigma}) = \bar{L}(\widetilde{\sigma}') \text{ and } X(\widetilde{\sigma}) = X(\widetilde{\sigma}').$$

Thus, the isomorphism $R_\sigma \simeq \bar{L}(\widetilde{\sigma}')/X(\widetilde{\sigma}')$ follows from Theorem 4.2. Hence, by Theorem 4.8, we have the claim. $\square$

Corollary 4.14. *Let $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ be given. If $i_{GM}(\sigma)$ is irreducible, then $i_{G'M'}(\sigma')$ is irreducible.*

The following theorem asserts that Knapp-Stein R-groups for G' are invariant on L-packets (cf. [Gol94a, Corollary 2.5] for G).

Theorem 4.15. *For any $\sigma'_1, \sigma'_2 \in \Pi_\phi(M')$, we have*

$$R_{\sigma'_1} \simeq R_{\sigma'_2}.$$

Proof. By fixing $\sigma \in \Pi_\phi(M)$, Lemma 4.11 shows $W'_{\sigma'_1} = W'_{\sigma'_2}$. It then suffices to show that $W(\sigma'_1) \simeq W(\sigma'_2)$. From Lemma 3.4, choose $x \in \widetilde{M}'$ such that $^x\sigma'_1 \simeq \sigma'_2$. We define a map

$$w' \mapsto x w' x^{-1}$$

from $W(\sigma'_1)$ to $W(\sigma'_2)$. Since M' is a normal subgroup of $\widetilde{M}'$, the element $xw'x^{-1}$ must be in $N_{\widetilde{G}'}(\mathbf{M}') \cap G' = N_{G'}(M')$, and $W(\sigma'_2)$. Note that this map is an injective homomorphism. In the same manner, we obtain another injective homomorphism

$$w' \mapsto x^{-1} w' x$$

from $W(\sigma'_2)$ to $W(\sigma'_1)$. It is clear that one map is the inverse of the other. Thus, the proof is complete. □

Corollary 4.16. *Let σ'_1, σ'_2 be given $\in \Pi_\phi(M')$. Then $i_{G'M'}(\sigma'_1)$ is irreducible if and only if $i_{G'M'}(\sigma'_2)$ is irreducible.*

Now we describe the difference between R_σ and $R_{\sigma'}$ (cf. Appendix A for another interpretation). We define a finite quotient

$$(4.3) \qquad\qquad W^*(\sigma') := W(\sigma)/W(\sigma')$$

from Proposition 4.9. So, due to Proposition 4.11 we have

$$(4.4) \qquad\qquad 1 \to R_{\sigma'} \to R_\sigma \to W^*(\sigma') \to 1,$$

which implies $W^*(\sigma') \simeq R_\sigma/R_{\sigma'}$. Given any σ'_1, $\sigma'_2 \in \Pi_\phi(M')$, we have from Theorems 4.2 and 4.15

$$(4.5) \qquad\qquad W^*(\sigma'_1) \simeq W^*(\sigma'_2).$$

Remark 4.17. We note that $W^*(\sigma')$ can be realized as the set $\{^w\sigma' : w \in W(\sigma)\}$. Further, given σ'_1 and σ'_2 in $W^*(\sigma')$, we have $JH(\sigma'_1) = JH(\sigma'_2)$. Here $JH(\sigma'_i)$ is the set of all irreducible constituents in $i_{G'M'}(\sigma'_i)$ for $i = 1, 2$.

Remark 4.18. We note that $W^*(\sigma')$ can be non-trivial, in which case $R_{\sigma'} \subsetneq R_\sigma$, as is discussed in example 6.3.4 of [CL12].

5. Multiplicities on F-inner forms of SL_n

In this section we discuss several possible multiplicities which occur in the restriction and the parabolic induction of an F-inner form $\mathbf{G}'$ of $\mathbf{G} = SL_n$. We express these in terms of cardinalities of L-packets the difference between R-groups of SL_n and its F-inner form. We continue with the notation in Section 4.2.

Throughout Section 5, we let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M} = \widehat{M}'$ be an elliptic tempered L-parameter. Let $\Pi_\phi(M)$ and $\Pi_\phi(M')$ be L-packets of M and M', respectively, associated to ϕ. Consider ϕ as an L-parameter of G and G' through a natural embedding $\widehat{M} \hookrightarrow \widehat{G} = \widehat{G'}$. We denote by $\Pi_\phi(G)$ and $\Pi_\phi(G')$ L-packets of G and G', respectively, associated to ϕ (see Section 3.1).

Let $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ be given. Choose $\widetilde{\sigma} \in \Pi_{\text{disc}}(\widetilde{M})$ and $\widetilde{\sigma}' \in \Pi_{\text{disc}}(\widetilde{M}')$ such that $\sigma \hookrightarrow \widetilde{\sigma}|_{M'}$ and $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$. We consider the following isomorphism

$$(5.1) \qquad\qquad (i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'} \simeq i_{G'M'}(\widetilde{\sigma}'|_{M'})$$

as G'-modules, since the restriction and the parabolic induction are compatible (cf. [BJ04, Proposition 4.1]). Fix an irreducible constituent π' of $i_{G'M'}(\sigma')$. We use the following notation.

- $\mathbf{m}(\pi')$ denotes the multiplicity of π' in $i_{G'M'}(\sigma')$.
- $\mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'})$ denotes the multiplicity of π' in the restriction $(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}$.
- $\mathbf{m}_{M',G'}(\widetilde{\sigma}'|_{M'})$ denotes the multiplicity of π' in $i_{G'M'}(\widetilde{\sigma}'|_{M'})$.

In what follows, we present these multiplicities in terms of $|W^*(\sigma')|$, $|\Pi_\phi(M)|$, $|\Pi_\phi(M')|$, $|\Pi_\phi(G)|$ and $|\Pi_\phi(G')|$.

Lemma 5.1. *We have*
$$\mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}) = \sqrt{\frac{|\Pi_\phi(G)|}{|\Pi_\phi(G')|}}.$$

Proof. Consider the commuting algebra
$$\mathrm{End}_{G'}(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}')) = \mathrm{Hom}_{G'}(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'), i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}')).$$

Applying [Tad92, Proposition 2.4], we have
$$\dim_{\mathbb{C}} \mathrm{End}_{G'}(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}')) = |\{\eta \in (\widetilde{M}/M)^D : i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \simeq i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}) \otimes \eta\}|$$
$$= |X(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|.$$

On the other hand, since the restriction $(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}$ is completely reducible, [Tad92, Lemma 2.1], we have a direct sum
$$(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'} \simeq \bigoplus_{\{\tau'\}} \mathbf{m}\, \tau'.$$

Here τ' runs through all irreducible inequivalent constituents in $(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}$, and $\mathbf{m}$ is the common multiplicity of irreducible constituents in $(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}$. We note that $\mathrm{Hom}_{G'}(\tau_1', \tau_2') = 0$ unless $\tau_1' \simeq \tau_2'$, in which case $\mathrm{Hom}_{G'}(\tau_1', \tau_2') \simeq \mathbb{C}$ by Schur's lemma. Hence, we get
$$\mathrm{End}_{G'}(\bigoplus_{\{\tau'\}} \mathbf{m}\, \tau') \simeq \bigoplus_{\{\tau'\}} \mathrm{End}_{G'}(\tau')^{\mathbf{m}^2} \simeq (\mathbb{C})^{\mathbf{m}^2}$$

(cf. [GK82, Lemma 2.1(d)]). We note that the set of all irreducible inequivalent constituent in $(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}$ equals the L-packet $\Pi_\phi(G')$ (see §3.1). Replacing the common multiplicity $\mathbf{m}$ by the multiplicity $\mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'})$ of π', we thus have
$$\dim_{\mathbb{C}} \mathrm{End}_{G'}(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}')) = [\mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'})]^2 \cdot |\Pi_\phi(G')|.$$

Note that $X(i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))$ equals $X(i_{\widetilde{G}\widetilde{M}}(\widetilde{\sigma}))$, which is in bijection with $\Pi_\phi(G)$ (Proposition 4.3). Therefore, we have

$$(5.2) \qquad \mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}) = \sqrt{\frac{|\Pi_\phi(G)|}{|\Pi_\phi(G')|}}.$$

This proves the lemma.   □

Lemma 5.2. *We have*
$$\mathbf{m}_{M',G'}(\widetilde{\sigma}'|_{M'}) = \sqrt{\frac{|\Pi_\phi(M)|}{|\Pi_\phi(M')|}} \cdot |W^*(\sigma')| \cdot \mathbf{m}(\pi').$$

Proof. Fix σ' in the restriction $\widetilde{\sigma}'|_{M'}$ such that π' is an irreducible constituent of $i_{G'M'}(\sigma')$. We note that
$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi', i_{G'M'}(\widetilde{\sigma}')) = m_1^2 m_2 m_3^2,$$

where

- m_1 denotes the common multiplicity of each constituent in $\widetilde{\sigma}'|_{M'}$,
- m_2 denotes the number of inequivalent components σ' in the restriction $\widetilde{\sigma}'|_{M'}$ such that π' is an irreducible constituent of $i_{G'M'}(\sigma')$, and
- m_3 denotes the multiplicity of π' in $i_{G'M'}(\sigma')$.

Thus, the multiplicity $\mathbf{m}_{M',G'}(\widetilde{\sigma}'|_{M'})$ equals $m_1^2 m_2 m_3^2$. In the same manner as equality (5.2), we have $m_1 = \sqrt{\frac{|\Pi_\phi(M)|}{|\Pi_\phi(M')|}}$. Further, from Remark 4.17, we have $m_2 = |W^*(\sigma')|$. We also have $m_3 = \mathbf{m}(\pi')$ by definition. Therefore, the proof is complete. $\square$

Proposition 5.3. *We have*

$$(5.3) \qquad \sqrt{\frac{|\Pi_\phi(G)|}{|\Pi_\phi(G')|}} = \sqrt{\frac{|\Pi_\phi(M)|}{|\Pi_\phi(M')|}} \cdot |W^*(\sigma')| \cdot \mathbf{m}(\pi').$$

Proof. From the isomorphism (5.1), we have

$$\mathbf{m}_{M',G'}((i_{\widetilde{G}'\widetilde{M}'}(\widetilde{\sigma}'))|_{G'}) = \mathbf{m}_{M',G'}(\widetilde{\sigma}'|_{M'}).$$

The proposition is thus a consequence of Lemmas 5.1 and 5.2. $\square$

Remark 5.4. Let π_1' and π_2' be irreducible constituents of $i_{G'M'}(\sigma_1')$ and $i_{G'M'}(\sigma_2')$ for some σ_1' and σ_2', respectively, in $\Pi_\phi(M')$. Due to (4.5), it turns out that all the factors except $\mathbf{m}(\pi')$ in equality (5.3) do not depend on the choice of π' . Hence, we have

$$\mathbf{m}(\pi_1') = \mathbf{m}(\pi_2').$$

Further, both ratios $\dfrac{|\Pi_\phi(G)|}{|\Pi_\phi(M)|}$ and $\dfrac{|\Pi_\phi(G')|}{|\Pi_\phi(M')|}$ are always square integers.

Appendix A. Actions of Characters on L-packets

Throughout Appendix A, we use the notation in Sections 2 and 3. In this appendix, we describe the difference between actions of characters of M and M' on their L-packets and interpret the difference in R-groups for SL_n and its F-inner forms in terms of characters (see (4.4) for another interpretation). We identify $\widetilde{M}/M$ and $\widetilde{M}'/M'$ (see Remark 2.5).

Throughout Section 4.2, we let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M} = \widehat{M}'$ be an elliptic tempered L-parameter. Let $\Pi_\phi(M')$ be an L-packet of M' associated to ϕ. Given $\sigma' \in \Pi_\phi(M')$, we define a set

$$X(\sigma') := \{\chi : M' \to S^1 \mid \sigma' \otimes \chi \simeq \sigma'\},$$

which consists of all unitary characters on M' stabilizing σ'. Here S^1 denotes the unit circle in $\mathbb{C}^\times$. It is easy to check that $X(\sigma')$ is an abelian group. We define a set

$$(A.1) \qquad X_{M'}(\phi) := \{\chi : M' \to S^1 \mid \sigma' \otimes \chi \in \Pi_\phi(M') \text{ for some } \sigma' \in \Pi_\phi(M')\},$$

which consists of all unitary characters on M' stabilizing $\Pi_\phi(M')$. For any $\sigma' \in \Pi_\phi(M')$, it is clear that $X_{M'}(\phi) \supseteq X(\sigma')$.

Lemma A.1. *The set $X_{M'}(\phi)$ is a finite abelian group.*

Proof. First, we claim that, given any character $\chi : M' \to S^1$,

(A.2) $\sigma'_0 \otimes \chi \in \Pi_\phi$ for some $\sigma'_0 \in \Pi_\phi(M') \iff \sigma' \otimes \chi \in \Pi_\phi$ for all $\sigma' \in \Pi_\phi(M')$.

Indeed, we choose $\widetilde{\sigma}'_0 \in \Pi_{\mathrm{disc}}(\widetilde{M'})$ such that $\sigma'_0 \hookrightarrow \widetilde{\sigma}'_0|_{M'}$. Let $\tilde{\chi} : \widetilde{M'} \to S^1$ be a character whose restriction to M' is identical with χ. From our construction of L-packets for M', the set of inequivalent irreducible constituents in $\widetilde{\sigma}'_0|_{M'}$ equals $\Pi_\phi(M')$. Since $\sigma'_0 \otimes \chi$ is an irreducible constituent of both $\widetilde{\sigma}'_0|_{M'}$ and $(\widetilde{\sigma}'_0 \otimes \tilde{\chi})|_{M'}$, we have

(A.3) $$\widetilde{\sigma}'_0|_{M'} \simeq (\widetilde{\sigma}'_0 \otimes \tilde{\chi})|_{M'} \simeq (\widetilde{\sigma}'_0|_{M'}) \otimes \chi$$

as representations of M'. Thus, it follows that $\sigma' \otimes \chi$ lies in Π_ϕ for any $\sigma' \in \Pi_\phi(M')$.

Next, we show that $X_{M'}(\phi)$ is an abelian group. Let χ_1 and χ_2 be in $X_{M'}(\phi)$. It suffices to show that $\chi_2 \chi_1^{-1} \in X_{M'}(\phi)$ since $X_{M'}(\phi)$ is a subset of the abelian group of unitary characters on M'. From the definition of $X_{M'}(\phi)$ we have σ'_1 and σ'_2 in $\Pi_\phi(M')$ such that both $\sigma'_1 \otimes \chi_1$ and $\sigma'_2 \otimes \chi_2$ lie in $\Pi_\phi(M')$. We set $\sigma'_* := \sigma'_1 \otimes \chi_1$. Since $\sigma'_* \otimes \chi_1^{-1} = \sigma'_1 \in \Pi_\phi(M')$, it follows from the claim (A.2) that

$$\sigma'_2 \otimes \chi_2 \chi_1^{-1} \in \Pi_\phi(M').$$

Hence, we have $\chi_2 \chi_1^{-1} \in X_{M'}(\phi)$.

Finally, it remains to show that $X_{M'}(\phi)$ is finite. Let $\chi \in X_{M'}(\phi)$ be given such that $\sigma' \otimes \chi \in \Pi_\phi(M')$. Note that all members in $\Pi_\phi(M')$ have the same central character which is the restriction of the central character of $\widetilde{\sigma}'$ to the center $Z(M')$ of M'. Here $\widetilde{\sigma}'$ is a lifting of σ' such that $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$. So, the character χ on M' is trivial on $Z(M')$, which implies χ is trivial on $Z(M') \cdot M'_{\mathrm{der}}$. Applying Galois cohomology (cf. the proof of Proposition 3.5), we have the exact sequence

$$1 \to Z(M'_{\mathrm{der}}) \to Z(M') \times M'_{\mathrm{der}} \to M' \to H^1(F, Z(\mathbf{M}'_{\mathrm{der}})).$$

Since $H^1(F, Z(\mathbf{M}'_{\mathrm{der}}))$ is finite, we notice that $Z(M') \cdot M'_{\mathrm{der}}$ has finite index in M'. It thus follows that $X_{M'}(\phi)$ is finite. $\square$

Remark A.2. We note that, if $\Pi_\phi(M')$ is a singleton, it then follows that $X(\sigma') = X_{M'}(\phi)$.

Lemma A.3. *For any $\sigma'_1, \sigma'_2 \in \Pi_\phi(M')$, we have*

$$X(\sigma'_1) = X(\sigma'_2).$$

In other words, for any character χ on M', $\sigma'_1 \simeq \sigma'_1 \otimes \chi$ if and only if $\sigma'_2 \simeq \sigma'_2 \otimes \chi$.

Proof. There exists an element $g \in \widetilde{M'}$ such that $\sigma'_2 \simeq {}^g\sigma'_1$ by Lemma 2.11 and our construction of $\Pi_\phi(M')$ in Section 3.1. Since $M'_{\mathrm{der}} \subseteq M' \subseteq \widetilde{M'}$, we notice that any character χ on M' is the restriction of a character $\widetilde{\chi}$ on $\widetilde{M'}$. So, we have ${}^g\chi(m') = \widetilde{\chi}(g^{-1}m'g) = \widetilde{\chi}(m') = \chi(m')$ for any $m' \in M'$. Thus we get

$\chi \in X(\sigma'_2) \iff \sigma'_2 \simeq \sigma'_2 \otimes \chi$

$\qquad \iff {}^g\sigma'_1 \simeq ({}^g\sigma'_1) \otimes \chi \iff {}^g\sigma'_1 \simeq {}^g(\sigma'_1 \otimes \chi) \iff \sigma'_1 \simeq \sigma'_1 \otimes \chi \iff \chi \in X(\sigma'_1).$

$\square$

Let $\Pi_\phi(M)$ be an L-packet of M associated to ϕ. Given $\sigma \in \Pi_\phi(M)$, we define two groups $X(\sigma)$ and $X_M(\phi)$ in the same manner as those for M'

$$X(\sigma) := \{\chi : M \to S^1 \mid \sigma \otimes \chi \simeq \sigma\},$$

$$X_M(\phi) := \{\chi : M \to S^1 \mid \sigma \otimes \chi \in \Pi_\phi(M) \text{ for some } \sigma \in \Pi_\phi(M)\}.$$

Then we have

(A.4) $$X(\sigma) = X_M(\phi).$$

Indeed, if $\sigma, \sigma \otimes \chi \in \Pi_\phi(M)$ then, by [Tad92, Theorem 1.2], their restrictions to M_{der} are equivalent, and hence, by multiplicity one of restriction from $\widetilde{M}$, to M, they must be isomorphic. We remark that this is also known in the case for L-packets for $GSp(4)$ [GT10, Proposition 2.2].

We notice the absence of the multiplicity one of restriction from $\widetilde{M}'$ to M' (cf. [Art06, p.215]). We establish the following proposition which is a different phenomenon from the split form.

Proposition A.4. *Let $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$ be given. Then we have*

$$X(\sigma') \subseteq X_{M'}(\phi) \overset{1-1}{\longleftrightarrow} X_M(\phi) = X(\sigma).$$

Proof. We recall any character $\tilde{\chi}$ on $\widetilde{M} = \prod_{i=1}^k GL_{n_i}(F)$ is of the form $\prod_{i=1}^k \tilde{\chi}_i \circ \det_i$, where each $\tilde{\chi}_i$ denotes a character on $F^\times$ and $\det_i$ denotes the determinant on $GL_{n_i}(F)$. Likewise, any character $\tilde{\chi}'$ on $\widetilde{M}' = \prod_{i=1}^k GL_{m_i}(D)$ is of the form $\prod_{i=1}^k \tilde{\chi}_i \circ \mathrm{Nrd}_i$, where each $\tilde{\chi}_i$ denotes a character on $F^\times$ and each Nrd_i denotes the reduced norm on $GL_{m_i}(D)$. Here D denotes a central simple division algebra of dimension d over F with $n = dm$, where $\sum_{i=1}^k n_i = n$ and $\sum_{i=1}^k m_i = m$. Under the local Jacquet-Langlands correspondence between $\Pi_{\mathrm{disc}}(M)$ and $\Pi_{\mathrm{disc}}(M')$, if $\widetilde{\sigma}$ corresponds to $\widetilde{\sigma}'$, then $\widetilde{\sigma} \otimes \tilde{\chi}$ corresponds to $\widetilde{\sigma}' \otimes \tilde{\chi}'$, since each $\tilde{\chi}_i \circ \det_i$ corresponds to $\tilde{\chi}_i \circ \mathrm{Nrd}_i$ (see Remark 2.7).

Let $\eta' \in X(\sigma')$ be given. We denote by $\tilde{\eta}' : \widetilde{M}' \to S^1$ a character which restricts to η' on M'. Then, using above arguments, we have

$$\eta' \in X(\sigma') \Leftrightarrow \sigma' \simeq \sigma' \otimes \eta' \Leftrightarrow \widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \tilde{\eta}' \Leftrightarrow \widetilde{\sigma} \simeq \widetilde{\sigma} \otimes \tilde{\eta} \Leftrightarrow \sigma \simeq \sigma \otimes \eta \Rightarrow \eta \in X(\sigma),$$

where η is a character on M such that $\tilde{\eta}|_M = \eta$. $\qquad\square$

Remark A.5. We note that the lifting $\tilde{\chi}$ of χ is uniquely determined up to a character on $\widetilde{M}/M$ ([Tad92, Corollary 2.5]) and the local Jacquet-Langlands correspondence is uniquely characterized by the character relation (see Proposition 2.6). Hence, the bijection in Proposition A.4 becomes an isomorphism and unique.

Remark A.6. Let $\Psi_{nr}(M)$ be the group of all unramified characters on M. Given $\sigma \in \Pi_\phi(M)$, we set $\mathrm{Stab}_{\Psi_{nr}(M)}(\sigma) := \{\psi \in \Psi_{nr}(M) \mid \sigma \otimes \psi \simeq \sigma\}$. Given $\sigma' \in \Pi_\phi(M')$, we define $\Psi_{nr}(M')$ and $\mathrm{Stab}_{\Psi_{nr}(M')}(\sigma')$ for the inner form M' in the same manner. Note that $\Psi_{nr}(M) \simeq \Psi_{nr}(M')$ (cf. [Roc09, Section 6]). Hence, Proposition A.4 implies that

$$\mathrm{Stab}_{\Psi_{nr}(M')}(\sigma') \subseteq \mathrm{Stab}_{\Psi_{nr}(M)}(\sigma)$$

for any $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$.

Proposition A.7. *Let $\sigma' \in \Pi_\phi(M')$ be given. Choose $\widetilde{\sigma}' \in \Pi_{\mathrm{disc}}(\widetilde{M}')$ such that $\sigma' \hookrightarrow \widetilde{\sigma}'|_{M'}$. Let w be an element in $W_{\widetilde{M}'} = W_{M'}$ such that $^w\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta$ for some $\eta \in (\widetilde{M}'/M')^D$. Then there exists a unitary character $\chi' \in M'^D$ such that*

$$^w\sigma' \simeq \sigma' \otimes \chi'.$$

Proof. It is enough to show that there exist an irreducible representation ρ' of M'_{der} such that $^{w}\rho'$ is a component of both $\sigma'|_{M'_{\mathrm{der}}}$ and $(^{w}\sigma')|_{M'_{\mathrm{der}}}$. We shall follow the idea on [Gol06, Lemma 2.6] (cf. [Gol94a, Lemma 2.3]).

We note that M'_{der} is of the form

$$SL_{m_1}(D) \times SL_{m_2}(D) \times \cdots \times SL_{m_k}(D).$$

Write $\widetilde{\sigma}' = \pi'_1 \otimes \pi'_2 \otimes \cdots \otimes \pi'_k$. Let ρ' be an irreducible constituent in $\sigma'|_{M'_{\mathrm{der}}}$. Then ρ' is isomorphic to $\rho' \simeq \rho'_1 \otimes \rho'_2 \otimes \cdots \otimes \rho'_k$ for some irreducible constituent ρ'_i in $\pi'_i|_{SL_{m_i}(D)}$. Suppose $w = w_1 w_2 \cdots w_t$ is the disjoint cycle decomposition of w by regarding $W_{\widetilde{M'}}$ as a subgroup of the group S_k of permutations on k letters. Without loss of generality, we assume that $w_1 = (1\,2\cdots j)$. Since $^{w}\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta$, we then have $\pi'_{i+1} \simeq \pi'_i \otimes \eta \simeq \pi'_1 \otimes \eta^i$, for $i = 1, 2, \cdots, j-1$, and $\pi'_1 \simeq \pi'_1 \otimes \eta^j$. (Here, by abuse of notation, η is regarded as a character on $GL_{m_1}(D)$.) Thus, for each $1 \le i \le j$, the representation ρ'_i is an irreducible constituent in $\pi_1|_{SL_{m_1}(D)}$. By Lemma 2.11, for each $1 \le i \le j-1$, there is an $a_i \in D^\times$, so that $\rho'_{i+1} = {}^{\delta(a_i)}\rho'_i$, where

$$\delta(a) = \begin{pmatrix} a & \\ & I_{m_1-1} \end{pmatrix}.$$

Let $a_j = (a_1\,a_2 \cdots a_{j-1})^{-1}$. Then $^{\delta(a_j)}\rho'_j = \rho_1$. Set

$$b_1 = \mathrm{diag}(\delta(a_1), \delta(a_2), \cdots, \delta(a_j), 1, 1, \cdots, 1).$$

Then we have $\det \circ \mathrm{Nrd}(b_1) = 1$, and

$$^{b_1}\rho' = \rho'_2 \otimes \cdots \otimes \rho'_j \otimes \rho'_1 \otimes \rho_{j+1} \otimes \cdots \otimes \rho_k = {}^{w_1}\rho'.$$

Similarly, for $i = 2, 3, \cdots, t$, we can find such a b_i such that $\det \circ \mathrm{Nrd}(b_i) = 1$ and $^{b_i}\rho' = {}^{w_i}\rho'$. Setting $b = \mathrm{diag}(b_1, \cdots, b_t, 1, \cdots, 1)$, we have $b \in M'$ and $^{b}\rho' \simeq {}^{w}\rho'$. Therefore, since $^{b}\sigma' \simeq \sigma'$, the irreducible representation $^{w}\rho'$ of M'_{der} must be in both restrictions $\sigma'|_{M'_{\mathrm{der}}}$ and $(^{w}\sigma')|_{M'_{\mathrm{der}}}$. This completes the proof. $\qquad\square$

We set

$$\bar{W} := \{w \in W_{M'} : {}^{w}\widetilde{\sigma}' \simeq \widetilde{\sigma}' \otimes \eta \text{ for some } \eta \in (\widetilde{M}'/M')^D\}.$$

Then Proposition A.7 allows us to define a map from $\bar{W}$ to $X_{M'}(\phi)/X(\sigma')$ by sending $w \mapsto \chi = \chi_w$. Therefore, given any $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$, we have from Proposition 4.9 and the definition of $W^*(\sigma')$ in (4.3)

(A.5) $\qquad R_\sigma/R_{\sigma'} \simeq W^*(\sigma') = W(\sigma)/W(\sigma') \simeq \bar{W}/W(\sigma') \hookrightarrow X_{M'}(\phi)/X(\sigma').$

Here we note that the isomorphism $\bar{W} \simeq W(\sigma)$ comes from Lemma 4.1 and the local Jacquet-Langlands correspondence (cf. proof of Proposition 4.9).

Remark A.8. Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M} = \widehat{M}'$ be an elliptic tempered L-parameter. Assume that the L-packet $\Pi_\phi(M')$ is a singleton. Then, by Remark A.2 and (A.5), we have $R_\sigma = R_{\sigma'}$.

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KWANGHO CHOIY, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078-1058, U.S.A.
   E-mail address: kwangho.choiy@okstate.edu

DAVID GOLDBERG, DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, U.S.A.
   E-mail address: goldberg@math.purdue.edu